

On the spin geometry of supergravity and string theory

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We summarize the main results of our recent investigation of bundles of real Clifford modules and briefly touch on some applications to string theory and supergravity.

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I. Introduction

Spinor fields are a crucial ingredient of supersymmetric theories such as superstring theory and supergravity. As such, a complete understanding of spin geometry in all dimensions and signatures for which such theories can be defined is of paramount importance for determining the weakest assumptions under which one can make global sense of various models. Virtually all supergravity Lagrangians currently in use are only locally defined, and the task of promoting them to complete formulations of the corresponding theories is affected by ambiguities in how the local formulas should be interpreted globally. Some of the most subtle issues of this type concern the global nature of spinorial fields, whose precise determination requires, in particular, that one identifies the weakest topological conditions which guarantee that a particular global interpretation is in fact allowed. The problem is compounded in string theory, which induces quantum corrections whose precise nature is sensitive to the global spin geometry of the target space-time (M, g) , D-brane world-volumes etc in various dimensions.

When seriously approaching the problem of giving global formulations of supergravity and string theories, one soon finds out that the questions: *What is the allowed global nature of spinor fields in string and supergravity theories and what are weakest conditions under which such theories can be defined ?* remains to be fully answered.

Locally, spinor fields in supergravity theories are usually considered as functions valued in a vector space S_0 acted on by locally-defined gamma matrices γ^μ associated to a vielbien (=local frame of the tangent bundle of the space-time M) e^μ . This amounts to taking S_0 to be a representation of the Clifford algebra $\text{Cl}(T_x M, g_x)$ (or sometimes of its even part $\text{Cl}^{\text{ev}}(T_x M, g_x)$) of the tangent space $T_x M$ at every point x of M . Since the formulation found in the literature is only local, the vielbein and in fact the entire Lagrangian density is defined only on a contractible open subset U of M and these Clif-

ford algebras are fibers of the restricted Clifford bundle $\text{Cl}(TU, g|_U)$ (or of its even sub-bundle $\text{Cl}^{\text{ev}}(TU, g|_U)$), both of which are topologically trivial bundles of unital and associative \mathbb{R} -algebras. This local set up does not determine uniquely the global nature of the spinor fields appearing in a putative global extension of the local Lagrangian densities found in the literature. A further complication concerns the treatment of reality conditions. Although the local structure of supergravity theories often requires that various spinor fields are real, the physics literature dealing with these situations usually starts by considering spinors as locally-defined functions valued in a complex Clifford representation which is then *realified* by using an appropriate antilinear conjugation operator, a generally non-canonical process that obscures the global properties of the spinors. In particular, it sometimes happens that the charge conjugation operator which one would wish to use in order to impose a reality condition on a complex spinor field *cannot* be defined globally as an appropriate section of the bundle of endomorphisms of a spinor bundle, which means that the usual approach found in the physics literature is often not appropriate for a global and general description of real spinor fields.

Globally understanding spinor fields requires that one fully characterizes the vector bundle over space-time of which they are sections. A simple way to globalize Clifford representations is to assume that the space-time admits a Spin structure Q . In that case, spinor fields can be defined as global sections of a vector bundle S over M which is associated to Q through some given representation of the spin group. Such bundles S carry a globally-defined Clifford multiplication which makes them into bundles of modules over the fibers of the Clifford bundle $\text{Cl}(TM, g)$ of space-time.

However, a spin structure has no immediate physical meaning and it is not clear in general that one is physically required to assume that S is a vector bundle associated to such a structure. What is physically important (and directly measurable) is not the spin structure Q itself but physical observables constructed using sections of the vector bundle S . For example, one generally *cannot* deduce that (M, g) must admit any spin structure solely from the fact that it admits a vector bundle with

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a globally-defined Clifford multiplication ! Careful reasoning along these lines shows that some supergravity theories (and especially those coupled to scalar and vector matter) can sometimes be defined globally *without* requiring that the space-time admits a spin structure. Therefore, one is naturally lead to ask what are the *weakest* conditions under which a given theory with a locally known Lagrangian density can be globally defined, and in how many globally-inequivalent ways.

In fact, Clifford multiplication is the minimal ingredient needed to construct a Dirac operator and hence to formulate the fermion kinetic action. Hence, one should take Clifford multiplication and not spin structures (or any of their known extensions) as the *a priori* fundamental ingredient needed to describe the physics of spinor fields. This reasoning suggests that one should define spinors not as sections of a vector bundle associated to some Spin structure, but as sections of an abstract vector bundle equipped with Clifford multiplication. At this point, one has a choice between *inner* and *outer* Clifford multiplications¹. The former is a morphism $TM \otimes S \rightarrow S$ obeying the Clifford property while the latter is a morphism $TM \otimes S \rightarrow S'$, where S' is a vector bundle which need not be isomorphic with S . In this note, we shall consider exclusively bundles endowed with inner Clifford multiplication, hence S will be a bundle of modules over the full Clifford bundle $\text{Cl}(TM, g)$ of space-time. The fiber S_p of S at every point p of M carries a representation of the algebra $\text{Cl}(T_p M, g_p)$. This gives a unital morphism of bundles of algebras $\gamma : \text{Cl}(TM, g) \rightarrow \text{End}(S)$, which we shall call the *structure morphism* of S . For technical reasons, we also require that γ be *weakly faithful*, which means that the restriction of γ to the vector bundle $TM \subset \text{Cl}(TM, g)$ is injective. For brevity of language, we define a *pinor bundle*² to be a weakly faithful bundle S of Clifford modules. This definition leads to a few natural questions:

- *Is every pinor bundle associated to a spin structure? If not, to what principal bundle is it associated ?*
- *What is the topological obstruction to existence of a pinor bundle on a pseudo-Riemannian manifold (M, g) of arbitrary signature (p, q) ?*

These questions were addressed in [1]. Regarding the first question, the results of loc. cit. show that, in general, real pinor bundles are associated not to spin structures but to more general spinorial structures, which we

call *real Lipschitz structures*, following previous work by T. Friedrich and A. Trautman [2] concerning the complex case. The second question was completely solved in [1] for so-called *elementary pinor bundles*, defined as those pinor bundles whose fiberwise Clifford representation is irreducible.

II. Real Lipschitz structures and their relation to real pinor bundles

Let (V, h) be a quadratic vector space which is isomorphic with each fiber of the tangent bundle (TM, g) . A representation $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$ of the Clifford algebra $\text{Cl}(V, h)$ in a finite-dimensional real vector space S_0 is called *weakly faithful* if the restriction of η to the subspace V of $\text{Cl}(V, h)$ is injective. The *real Lipschitz group* $L(\eta)$ of η is the group consisting of all invertible operators a acting in S_0 whose adjoint action preserves the subspace $\eta(V)$ of $\text{End}_{\mathbb{R}}(S)$:

$$L(\eta) \stackrel{\text{def}}{=} \{a \in \text{Aut}_{\mathbb{R}}(S_0) | \text{Ad}(a)(\eta(V)) = \eta(V)\} \quad .$$

The *vector representation* of $L(\eta)$ is the group morphism $\text{Ad}_0 : L(\eta) \rightarrow \text{O}(V, h)$ defined through:

$$\text{Ad}_0(a) \stackrel{\text{def}}{=} (\eta|_V)^{-1} \circ \text{Ad}(a)|_{\eta(V)} \circ (\eta|_V) \quad .$$

A *real Lipschitz structure of type η* on (M, g) is an Ad_0 -reduction (Q, τ) of the principal bundle $P(M, g)$ of pseudo-orthogonal frames of (TM, g) , i.e. a pair formed of a principal $L(\eta)$ -bundle Q over M and an Ad_0 -equivariant fiber bundle map $\tau : Q \rightarrow P(M, g)$. A bundle (S, γ) of Clifford modules over (M, g) is weakly-faithful iff each fiberwise Clifford representation $\gamma_p : \text{Cl}(T_p M, g_p) \rightarrow \text{End}(S_p)$ (where $p \in M$) is weakly-faithful. Since M is connected, all fiberwise Clifford representations γ_p are *unbasedly* isomorphic³ with each other and hence with some fiducial weakly faithful Clifford representation $\eta : \text{Cl}(V, h) \rightarrow \text{End}_{\mathbb{R}}(S_0)$, where S_0 is a vector space which models the fiber of S . The representation η (considered up to unbased isomorphism of representations) is called the *type* of (S, γ) . One has the following key result:

Theorem 1. [1]. There exists an equivalence of categories between the groupoid of real Lipschitz structures of type η and the groupoid of real pinor bundles of type η defined over (M, g) . In particular, the underlying vector bundle S of every real pinor bundle (S, γ) of type

¹ Outer Clifford multiplication arises, for example, in the theory of Pin structures, in which situation it sometimes allows one to define a “modified” Dirac operator [11].

² The word “pinor” refers to the fact that we consider bundles of modules over the fibers of $\text{Cl}(TM, g)$ rather than over the fibers of $\text{Cl}^{\text{ev}}(TM, g)$.

³ This means that they are isomorphic in a certain category which is defined in [1] and which has more morphisms than the usual category of representations.

η is associated to the principal bundle Q of a Lipschitz structure (Q, τ) which has type η .

The theorem implies that (M, g) admits a real pinor bundle of type η iff it admits a real Lipschitz structure of type η , and that the classifications of these two kinds of objects up to the corresponding notion of isomorphism agree.

One can show that any irreducible real Clifford representation is weakly-faithful and that all such representations of $\text{Cl}(V, h)$ belong to the same *unbased* isomorphism class, which is determined by the signature (p, q) of (V, h) . A real pinor bundle (S, γ) is called *elementary* if its fibers are *irreducible* as real Clifford representations, which amounts to the requirement that its type η is irreducible. The real Lipschitz groups of irreducible Clifford representations are called *elementary*, as are the real Lipschitz structures whose type is given by such representations. For each quadratic vector space (V, h) , there exists an essentially unique elementary real Lipschitz group L , determined up to isomorphism by the signature (p, q) of (V, h) . Moreover, the nature of this group depends only on $p - q \pmod{8}$. One can show that $L = \mathbb{R}_{>0} \times \mathcal{L}$, where \mathcal{L} is a natural subgroup called the *reduced Lipschitz group*, which can be constructed using the so-called “Lipschitz norm”. Elementary real Lipschitz groups were classified in [1]. The result is summarized in Table II. A *reduced elementary Lipschitz structure* is defined like a Lipschitz structure, but using the group \mathcal{L} (and the restriction of Ad_0 to \mathcal{L}) instead of L . The groupoid of elementary real Lipschitz structures is equivalent with that of reduced elementary real Lipschitz structures, so the latter is also equivalent with the groupoid of elementary real pinor bundles. When $pq \neq 0$, \mathcal{L} is neither compact nor connected.

$p - q \pmod{8}$	\mathcal{L}	$\mathfrak{G}(p, q)$
0, 2	$\text{Pin}(p, q)$	1
3, 7	$\text{Spin}^o(p, q) \stackrel{\text{def}}{=} \text{Spin}(p, q) \cdot \text{Pin}_2^{\alpha_{p,q}}$	$\text{O}(2, \mathbb{R})$
4, 6	$\text{Pin}^q(p, q) \stackrel{\text{def}}{=} \text{Pin}(p, q) \cdot \text{Sp}(1)$	$\text{SO}(3, \mathbb{R})$
1	$\text{Spin}(p, q)$	1
5	$\text{Spin}^q(p, q) \stackrel{\text{def}}{=} \text{Spin}(p, q) \cdot \text{Sp}(1)$	$\text{SO}(3, \mathbb{R})$

TABLE I. Reduced elementary Lipschitz groups in signature (p, q) . The sign factor $\alpha_{p,q}$ equals -1 when $p - q \equiv 3$ and $+1$ when $p - q \equiv 7$ and we use the notation $\text{Pin}_2^+ \stackrel{\text{def}}{=} \text{Pin}(2, 0)$ and $\text{Pin}_2^- \stackrel{\text{def}}{=} \text{Pin}(0, 2)$. The last column lists the characteristic group. The symbol “ \cdot ” denotes direct product of groups divided by a central \mathbb{Z}_2 subgroup.

It is clear from this table that the conditions for existence of an elementary Lipschitz structure are generally *weaker* (and sometimes considerably so) than those for

existence of a spin structure. Every elementary Lipschitz group has a so-called *characteristic representation*, which is naturally associated to it as explained in [1]. The image of this representation is the so-called *characteristic group* $\mathfrak{G}(V, h)$, whose isomorphism type is listed in the last column of Table II. Accordingly, an elementary Lipschitz structure Q induces a principal *characteristic bundle* E (with structure group $\mathfrak{G}(p, q)$), which is associated to Q through the characteristic representation of the corresponding Lipschitz group; this bundle can be non-trivial only when $p - q \not\equiv 0, 1, 2$. For $p - q \equiv 0, 1, 2$, a Lipschitz structure is either a Spin or Pin structure and hence is of the classical type studied for example in [12]. When $p - q \equiv 5$, it is a Spin^q structure in general signature; the positive-definite case ($q = 0$) of such was studied in [13]. The cases $p - q \equiv 4, 6$ lead to Pin^q structures, which are a slight extension of Spin^q structures to non-orientable manifolds. The cases $p - q \equiv 3, 7$ lead to what we call Spin^o structures (which appear to be new).

The characteristic bundle of a Spin^o -structure is a principal $\text{O}(2)$ bundle, which suggests that it may be relevant to situations where spinors are *charged* under a $\text{O}(2)$ gauge group rather than under a $\text{U}(1)$ group. This fact may be relevant to understand the worldvolume theories of non-orientable D-branes. Let:

$$\begin{aligned} \sigma &:= \sigma_{p,q} \stackrel{\text{def}}{=} (-1)^{q + [\frac{d}{2}]} = \begin{cases} (-1)^{\frac{p-q}{2}} & \text{if } d = \text{even} \\ (-1)^{\frac{p-q-1}{2}} & \text{if } d = \text{odd} \end{cases} = \\ &= \begin{cases} +1 & \text{if } p - q \equiv 4, 0, 1 \\ -1 & \text{if } p - q \equiv 2, 3 \end{cases} \end{aligned}$$

Let $w_1^\pm(M)$ be the modified Stiefel-Whitney classes of (M, g) introduced in [12]; these classes depend on g but we don’t indicate this in the notation. The topological obstructions to existence of elementary real Lipschitz structures (and hence of elementary real pinor bundles) on (M, g) are as follows [1]:

- In the normal simple case ($p - q \equiv 0, 2$), (M, g) admits an elementary real pinor bundle iff $(M, -\sigma g)$ admits a Pin structure, which requires that the following condition is satisfied:

$$w_2^+(M) + w_2^-(M) + w_1^+(M)^2 + w_1^-(M)w_1^+(M) = 0.$$

- In the complex case ($p - q \equiv 3, 7$), (M, g) admits an elementary real pinor bundle iff it admits a Spin^o -structure, which happens iff there exists a principal $\text{O}(2, \mathbb{R})$ -bundle on M such that the following two conditions are satisfied:

$$w_1(M) = w_1(E)$$

$$w_2^+(M) + w_2^-(M) = w_2(E) + w_1(E)(pw_1^+(M) + qw_1^-(M)) \\ + \left[\delta_{\alpha, -1} + \frac{p(p+1)}{2} + \frac{q(q+1)}{2} \right] w_1(E)^2 ,$$

where $\alpha \stackrel{\text{def}}{=} \alpha_{p,q}$.

- In the quaternionic simple case ($p - q \equiv_8 4, 6$), (M, g) admits an elementary real pinor bundle iff $(M, -\sigma g)$ admits a Pin^q -structure, which happens iff there exists a principal $\text{SO}(3, \mathbb{R})$ -bundle E on M such that the following condition is satisfied:

$$w_2^+(M) + w_2^-(M) + w_1^\sigma(M)^2 + w_1^-(M)w_1^+(M) = w_2(E) .$$

- In the normal non-simple case ($p - q \equiv_8 1$), (M, g) admits an elementary real pinor bundle iff it admits a Spin structure, which requires that the following two conditions are satisfied:

$$w_1(M) = 0, \quad w_2^+(M) + w_2^-(M) = 0 .$$

- In the quaternionic non-simple case ($p - q \equiv_8 5$), (M, g) admits an elementary real pinor bundle iff it admits a Spin^q -structure, which happens iff there exists a principal $\text{SO}(3, \mathbb{R})$ -bundle E over M such that the following conditions are satisfied:

$$w_1(M) = 0, \quad w_2^+(M) + w_2^-(M) = w_2(E) .$$

III. Applications to string theory and supergravity

The results of reference [1] can be applied to study the spinorial structures needed to formulate various supergravity theories. In this section, we sketch a simple application to M-theory, obtaining a no-go result regarding the global interpretation of its spinor fields.

Consider M-theory on an eleven-dimensional Lorentzian manifold of “mostly plus” signature $(p, q) = (10, 1)$. The low energy limit is given by eleven-dimensional supergravity, whose supersymmetry generator is a 32-component real spinor ϵ . The gravitino Killing spinor equation contains terms with an odd number of gamma matrices acting on ϵ , implying that the whole Clifford algebra $\text{Cl}(T_x M, g_x)$ at a point $x \in M$ must act on the value of ϵ at x . If one assumes that ϵ is a global section of a vector bundle S endowed with *inner* Clifford multiplication, it follows that each fiber S_p must carry a real irreducible representation of $\text{Cl}(T_p M, g_p)$ and hence that S is an elementary real pinor bundle. Since $p - q = 9 \equiv_8 1$, we are in the normal simple case. Hence (M, g) admits an elementary real pinor bundle S if and only if it is oriented and spin. Since $w_2^-(M) = 0$, the corresponding topological

obstruction can be written as follows:

$$w_1^+(M) = w_1^-(M), \quad w_2^+(M) = 0 .$$

We conclude that, in signature $(10, 1)$, the supersymmetry parameter can be interpreted as a global section of an elementary real pinor bundle iff the space-time is orientable and spin.

Of course, M-theory can in fact be defined on Lorentzian eleven-manifolds admitting a Pin structure [4, 5], but that construction involves a bundle with external Clifford multiplication, which leads to a modified Dirac operator as in [11].

IV. Future directions

The results of [1] open up various directions for further research. Here are some questions which may be worth pursuing:

- Reference [1] classifies bundles of irreducible modules over $\text{Cl}(M, g)$. It would be interesting to classify bundles of faithful real Clifford modules over $\text{Cl}(M, g)$ and irreducible or faithful real Clifford modules over the even sub-bundle $\text{Cl}^{\text{ev}}(M, g)$, since such bundles may also be relevant in string theory and supergravity.
- It would be interesting to study the index theorem for general bundles of real Clifford modules, without assuming that (M, g) is spin.
- One could consider extending Wang’s classification [6] beyond the case of spin manifolds, characterizing manifolds admitting sections of an elementary real pinor bundle which are parallel with respect to a connection lifting the Levi-Civita connection on (M, g) and a fixed connection on the characteristic bundle.
- Killing and generalized Killing spinors were studied in the literature [7–9], usually on manifolds carrying a fixed Spin or Spin^c structure. Using our results, this could be extended to the most general pseudo-Riemannian manifolds admitting elementary real pinor bundles.
- One could apply our results to the spin geometry of branes in string and M-theory. As shown in reference [10], the worldvolume of orientable D-branes in the absence of H -flux admits a Spin^c -structure. In the unorientable case, this may become a Lipschitz structure.
- Our results may be useful to globally characterize the local spinor bundles appearing in exceptional generalized geometry [14], obtaining the topological obstructions to their existence.

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